Imagine a one dimensional box with impenetrable walls (potential infinite) at $x=0$ and $x=a$. The particle must be in the box since the walls are infinitely high, so the wave function $\psi = 0$ unless $0 < x < a$. Since the wave function must be continuous, it must therefore be zero also at the boundaries, $x = 0$ and $x = a$. These are called boundary conditions. Between the walls there is no force, zero potential energy, and so Schrodinger's equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \,,$$

which we rewrite as

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \,.$$

Now, we have to be able to solve this differential equation for $\psi$. This is a second order differential equation so there should be two linearly independent solutions. This is the most famous of all differential equations, called the harmonic oscillator equation. What are its solutions? Ask yourself what function, when differentiated twice gives itself back but with a minus sign? It must be either a sine or a cosine:

$$\begin{aligned}
&\frac{d^2y(x)}{dx^2} = -y(x) \\
&y(x) = A\sin x \\
&or \\
&y(x) = B\cos x
\end{aligned}$$

where $A$ and $B$ can be any constants. The most general solution of the equation is

$$y(x) = A\sin x + B\cos x \,.$$

So, for our Schrodinger equation, the only minor complication is the $k$. You should be able to convince yourself that the general solution for the wave function is $\psi(x) = A\sin(kx) + B\cos(kx)$. Now, we must find the constants $A$ and $B$. We use the boundary condition that $\psi(0) = 0$ to determine that $B$=0 because $\cos(0) = 1$, *i.e.* the wave function could not be zero at the origin unless $B$=0. We will worry about what $A$ is later. So now the wave function is $\psi(x) = A\sin(kx)$. Now, how are we going to satisfy the other boundary condition, namely that $\psi(a) = 0$? What must be true is that $\sin(ka) = 0$. But, the sine function is zero only if its argument is $\pi, 2\pi, 3\pi, 4\pi, \ldots$ So we can write

$$\psi(a) = 0 \Rightarrow ka = n\pi \quad n = 1, 2, 3, \ldots$$

So, this second boundary condition places a condition on $k$. But $k = \sqrt{\frac{2mE}{\hbar^2}}$, so only certain energies are allowed which satisfy the boundary conditions; I will label these as $E_n$:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \,.$$

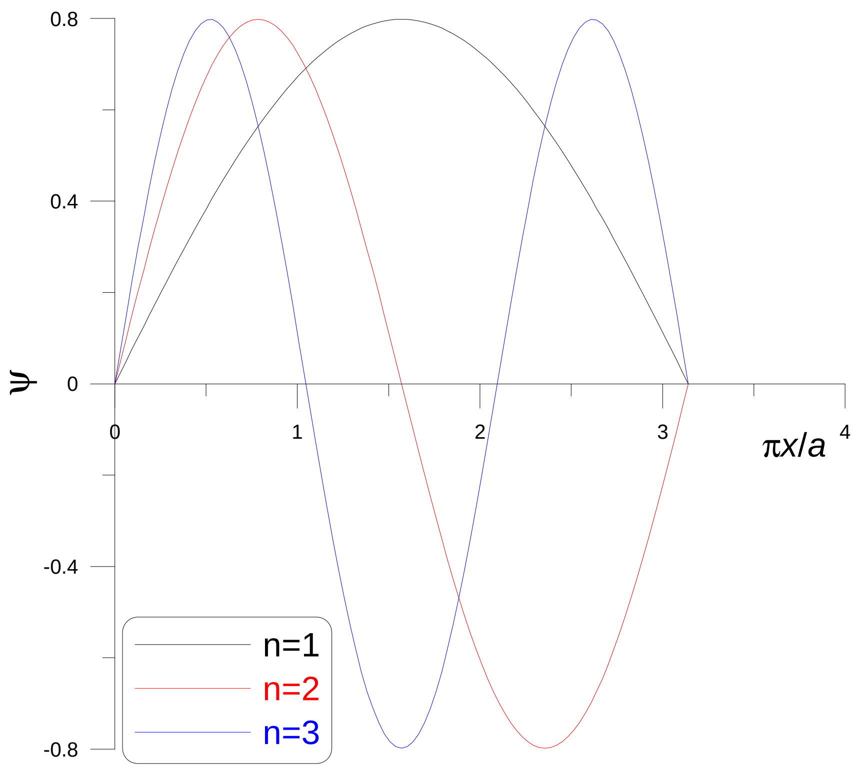
So, for each $E_n$ there is a corresponding wave function $\psi_n$ which will have an associated constant $A_n$. So, finally, we need to get the constants $A_n$ so that we can find the wave function of the $n^{\text{th}}$ state with no arbitrary constants. We do this using a procedure called normalization. The meaning of a wave function is that its square represents the probability distribution, that is $\psi^2(x)dx$ is the probability that the particle is between $x$ and $x+dx$. Since the probability of finding the particle anywhere between $x=0$ and $x=a$ must be unity, that is

$$\int_0^a \psi_n^{\,2}(x)dx = 1 = A_n^{\,2}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx \ ,$$

we can deduce $A_n$ by evaluating the integral and solving. The integral is easily shown to be $\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{a}{2}$ and so $A_n = \sqrt{\frac{2}{a}}$. Note that the coefficients $A_n$ are independent of $n$ which just means we didn't need to subscript them in the first place; it just works out that way for this particular problem but in general the normalization coefficients will depend on the energy level. Finally we can write the wave function for the $n^{\text{th}}$ state whose energy is $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right).$$

The first three wave functions are plotted below. (Note that the point $\pi$ on the axis corresponds to $x=a$.)



It is also instructive to plot $\psi_n^{\ 2}$.